

# Amplitude equations for wave packets in slightly inhomogeneous unstable flows

By T. F. BALSÀ

Department of Aerospace and Mechanical Engineering, University of Arizona,  
Tucson, AZ 85721, USA

(Received 17 August 1987 and in revised form 28 December 1988)

We derive several versions of the (complex) amplitude equation of an inviscid wave packet travelling on a slightly inhomogeneous (and possibly unsteady and viscous) *unstable* base flow. This is done with complete generality, without any reference to the dimensions of physical and propagation spaces, by using the usual high-frequency ansatz. The final results are extremely simple: volume integrals of a complex wave action density are conserved subject to an appropriate flux and a source term. The latter is expressible in a remarkably concise way in terms of the gradient of the base flow acceleration and vanishes when the base flow is inviscid. The simplicity of our results hinges on a transformation of the dependent variables and on a suitable decomposition of these in cross- and propagation spaces. Our results are also discussed with the help of three different Lagrangian densities and their associated kinematic wave theories which are based on a basic identity due to Hayes.

---

## 1. Introduction

There is considerable renewed interest in the study of the evolution of small-amplitude disturbances in an unstable base flow (such as a mixing layer). Historically, such studies were aimed at understanding the laminar–turbulent transition process; more recently, however, some of the focus has shifted to fully turbulent flows. When these flows are artificially perturbed by certain types of external excitation, so-called large-scale structures are produced. The characteristics of these structures are actively being studied, both experimentally and theoretically.

For the purposes of this paper, it is sufficient to note that many of the important features of the large-scale structures can be described, with *surprising* and remarkable accuracy, by the linear instability modes of a (fictitious) base flow, whose velocity profile is the long-time average velocity. This remark has been confirmed by several independent studies in a variety of free-shear flows (Gaster, Kit & Wignanski 1985 (in mixing layers); Wignanski, Champagne & Marasli 1986 (in wakes); Petersen & Samet 1988 (in jets)). The excitation of turbulent flows is desirable from the point of view of controlling the dynamics of the flow via the introduction of *controlled* large-scale structures; for example, the size of the separation zone on the suction side of an airfoil may be reduced by irradiating this zone with sound (I. Wignanski 1986, private communication).

An arbitrary small disturbance in a typical unstable flow will quickly ‘disperse’ into an instability wave packet (Gaster & Grant 1975). Here, the physical mechanism of dispersion (perhaps a more convenient term is distortion) is quite different from classical wave dispersion; for the case of instability waves, dispersion arises from variations of growth rate with respect to wavenumber in addition to variations of

phase speed. Since the overall characteristic lengthscale of the packet is usually a good bit smaller than the lengthscale on which the base flow appears non-parallel (owing, for example, to viscous or turbulent stresses), the elementary oscillations in the packet are nearly plane, progressive modal waves whose characteristics (e.g. frequency, growth rate, wavenumber) are changing slowly in some sense (to be made precise in §4). These remarks suggest that the methods of multiple scales and kinematic wave theory are ideally suited for the description of wave packets in unstable shear flows.

Indeed, both of these methods have been applied, beginning with the pioneering contributions of Crighton & Gaster (1976) and Landahl (1972). In the former, the authors use the method of multiple scales to obtain the lowest-order uniformly valid solution for a spatial instability mode evolving on a slowly diverging jet. Their small parameter is the ratio of the streamwise wavelength of the mode to the streamwise lengthscale of the flow; we call this parameter (or any other reasonable representation of it) the inhomogeneity parameter,  $\epsilon \ll 1$ . On the other hand, Landahl (1972) proposes an extension to Whitham's (1965) 'amplitude equation' for non-conservative systems (i.e. instability waves) and uses his equation to discuss a possible mechanism for the 'breakdown' of laminar flows. This breakdown arises when the 'energy' of the wave, carried by the group velocity, is propagating along a ray tube whose cross-sectional area decreases to (near) zero (i.e. a simple focus); under certain simplifying conditions, this happens when the difference between the phase velocity of the primary wave and the (real) group velocity of the secondary wave vanishes. In other words, in a reference frame moving with the phase velocity of the primary wave, the energy of the secondary wave cannot propagate away (because the group velocity is zero); in a sense, the waves 'pile up' and produce large fluctuations which result in breakdown. Some of the original criticisms of Stewartson (1974) on this issue have been partially removed by Nayfeh (1980) and Russell (1986) (see also Landahl 1982 and Chin 1980).

At this point, we must mention two important contributions to kinematic wave theory by Whitham (1965) and Hayes (1970). In a seminal paper, Whitham defined wave action density and flux in terms of derivatives, of an *averaged Lagrangian*, with respect to frequency and wavenumber, and showed that wave action obeys a conservation law. The averaged Lagrangian is obtained by substituting into the actual Lagrangian density an elementary progressive wave solution with slowly varying amplitude,  $A$ , wavenumber,  $k$ , and frequency,  $\omega$ , and then averaging the resultant equation over one oscillation (while ignoring any changes in these slowly varying quantities). If we now define a local (slowly varying) phase,  $\phi$ , whose suitable time and space derivatives give the local frequency and wavenumber, then the original variational principle, now applied to the averaged Lagrangian (which is written in terms of the derivatives of the phase), yields the conservation law for the wave action. This law arises from the Euler equation corresponding to variations in the phase. The enormous beauty of Whitham's approach is its power to deal simply with progressive waves in nonlinear (as well as linear) conservative systems (e.g. it enables one to naturally define dispersion for nonlinear systems).

In an equally important paper, Hayes (1970) proposed an alternative approach to the definition of wave action. In Whitham's (1965) theory, this entity arises from a variational principle applied to an *averaged Lagrangian*, while in Hayes' work, we find an absolute conservation law, valid in physical space, defined over a periodic one-parameter family of solutions (see equations (9c) of this paper). These two approaches differ for modal waves; Hayes' approach gives a conservation law in

physical space, and the integral of this law over cross-space gives Whitham's results. In §7, we shall apply Hayes' method to the problem of instability wave packets at hand.

The concepts of wave action density and flux may be generalized considerably; these generalizations do not depend on the approximations of slow amplitude modulation, linearization, and conservative motion, or on the existence of a Lagrangian for the description of the flow field (Andrews & McIntyre 1978*a, b*). The essential idea is the separation of the base flow and the disturbance (i.e. wave) field as cleanly as possible – even at finite amplitudes; this is accomplished most readily in terms of a displacement variable and a hybrid (Eulerian–Lagrangian) formulation. Andrews & McIntyre find (as do we) that wave action is not conserved when the motion is non-conservative.

In spite of the power of the approaches of Whitham (1965) and Hayes (1970), they are seldom applied to instability waves because of a number of obstacles (to be discussed below). Notable exceptions are the works of Landahl (1972, 1982) and Russell (1986). Wave packets and spatial instability modes in inhomogeneous base flows are usually treated by the method of multiple scales. Itoh (1980) obtains an amplitude equation for a two-dimensional wave packet in a parallel shear flow and finds that the square of the complex amplitude,  $A$ , obeys the usual conservation law. The situation is unreasonably more complicated, however, when the base flow is slowly changing; although, in this case, it is possible to obtain an amplitude equation (for  $A^2$ ) which contains a 'source term', proportional to  $A^2$ . Even in the simplest case, the coefficient of this term is horrendously lengthy and complicated; it involves the partial derivatives of the modes with respect to the slowly varying coordinates, as well as the wavenumber. These derivatives are taken in an augmented space implied by (23*b*). Of course, derivatives of the base flow also appear.

Although the relevant amplitude equation *can* be obtained in a fairly straightforward manner (see Nayfeh 1980; Itoh 1981) and *can* be used in numerically oriented studies to obtain answers to specific problems, it seems almost certain that the essential physics is contained in this equation in a very awkward manner. For example, it is far more important to know that there is an adiabatic invariant for the simple pendulum whose length is slowly changing than to know that there is an amplitude equation which, in principle, may be solved to obtain useful results.

Part of the problem is that the (Eulerian) perturbation velocity,  $\mathbf{u}$ , and perturbation pressure,  $p$ , are not the best variables to use for the derivation of an amplitude equation. After all, even in conservative wave systems, the law of wave action 'naturally falls out' only from a Lagrangian formalism; the derivation of this by other means is a difficult task (Luke 1966; Bretherton 1968). Eulerian velocity fields are seldom used in the construction of a Lagrangian density.

Russell (1986), clearly realizing some of these problems, introduced a new set of dependent variables, say  $(\alpha, p)$ , in place of  $(\mathbf{u}, p)$ . In addition, he also introduced auxiliary variables, say  $(\beta, \pi)$ , and constructed a (bilinear) Lagrangian density in the eight variables  $\Phi = (\alpha, \beta, p, \pi)$ . By choosing  $(\alpha, p)$  and  $(\beta, \pi)$  to be unstable and stable waves, respectively, Russell could average the Lagrangian density over one oscillation and then, building on the ideas of Whitham (1965), he derived a pure conservation law for the (bilinear) wave action.

The introduction of a bilinear Lagrangian density is desirable (but not absolutely necessary if one is willing to deal with not only the lowest-order solution but also the next higher-order one) since it overcomes the presence of rapidly varying terms in the averaged Lagrangian (see §7.2). On the other hand, the limitation of the bilinear

wave action is that it only says something (very roughly speaking!) about the product amplitude ( $AB$ ), where  $A$  and  $B$  are the amplitudes of  $(\alpha, p)$  and  $(\beta, \pi)$ , respectively. From this, it is generally not possible to discern the individual behaviour of  $A$  (or  $B$ ). In some sense, the bilinear wave action is reminiscent of the 'adjoint wave action density' of Nayfeh (1980).

However, by far the most serious limitation of Russell's (1986) work is his assumption that the base flow is inviscid. In most practical flows of interest, the (small) inhomogeneity of the base flow is caused by viscous or turbulent stresses (i.e. diverging mixing layers, jets, and wakes), and one would certainly like to trace the evolution and secondary instabilities of a wave packet in these flows. This problem, not addressed by Russell, is important both for the laminar-turbulent transition process as well as for the control of turbulent flows by external excitation.

Our principal objective is to develop a simple amplitude equation for a completely arbitrary wave packet riding on a slightly inhomogeneous, but otherwise arbitrary, base flow. There is no restriction on the dimensionality of physical space or that of propagation space. The base flow is quite general and may very well be viscous or turbulent, although the wave packet is assumed to be inviscid. Our success depends on the introduction of a new set of dependent variables, say  $(\alpha, p)$ , and on the decomposition of  $\alpha$  into its components in cross- and propagation spaces. Some of the basic ideas are outlined in §2. Section 3 contains a discussion of the slowly changing nature of the base flow and some additional in-depth comments on the work of Russell.

In §§4, 5, and 6, we derive two versions of the amplitude equation, (32c) and (35), using the usual high-frequency ansatz, and find that ordinary wave action is *not* conserved in the usual sense; there is a single source term which depends on the gradient of the base flow acceleration when the base flow has dissipation.

In §7, we discuss the bilinear Lagrangian using the formalism of Hayes (1970). It is interesting to note that the bilinear wave action *is* conserved, although the ordinary one is not. This simply reinforces the fact that the conservation of the bilinear wave action generally does not say much about the development of an unstable wave packet in a flow. Thus, the limitation of Russell's (1986) work is immediately apparent. We also give a very explicit representation of the bilinear wave action density in terms of the relevant amplitudes and mode shapes (45a). Finally, we show how the amplitude equation may be obtained from an intrinsic Lagrangian density defined entirely in terms of quantities of physical interest, without the introduction of auxiliary variables  $(\beta, \pi)$ .

## 2. Formulation of the problem

Consider an incompressible (i.e. constant-density) unperturbed base flow, of velocity  $U = U(\mathbf{x}, t)$ , obeying the equations of momentum and continuity (suitably non-dimensionalized),

$$\frac{DU}{Dt} = \mathcal{F}(\mathbf{x}, t), \quad (1a)$$

$$\nabla \cdot U = 0, \quad (1b)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \cdot \nabla \quad (1c)$$

denotes differentiation in time, following the unperturbed fluid motion. A point in physical space is represented by the Cartesian vector  $\mathbf{x}$ , and  $t$  stands for time. In this

paper, the momentum equation (1a) has a dual interpretation. In the first interpretation, which is a kinematic one,  $\mathcal{F}(\mathbf{x}, t)$  is simply a shorthand notation for the fluid acceleration, whereas in the second interpretation, which is a dynamic one,  $\mathcal{F}(\mathbf{x}, t)$  stands for the applied force. If the base flow is laminar,  $\mathcal{F}(\mathbf{x}, t)$  is readily expressible in terms of the (unperturbed) pressure gradient, the divergence of the shear stress, and (possibly) the body forces acting on the fluid. On the other hand, if the base flow is turbulent, we choose  $\mathbf{U}(\mathbf{x}, t)$  to be the long-time average velocity (in the spirit of Gaster *et al.* 1985). In this latter case, no explicit closed-form expression can be given for  $\mathcal{F}(\mathbf{x}, t)$  if the quantity is interpreted as a force. Unless otherwise stated, we adhere to the first interpretation, which is more general than the second, although our final results assume an especially simple form when  $\mathcal{F}(\mathbf{x}, t)$  consists of the unperturbed pressure gradient only ( $P$  denotes the unperturbed pressure). In this paper, we consider  $\mathbf{U} = \mathbf{U}(\mathbf{x}, t)$  to be a *given* quantity so that  $\mathcal{F} = \mathcal{F}(\mathbf{x}, t)$  can always be computed from (1a).

Let us now assume that our base flow is perturbed by an arbitrary disturbance, whose (perturbation) velocity and pressure fields are denoted by  $\mathbf{u}(\mathbf{x}, t)$  and  $p(\mathbf{x}, t)$ , respectively. The relevant linearized equations for these quantities are

$$\frac{D\mathbf{u}}{Dt} + \mathbf{u} \cdot \nabla \mathbf{U} = -\nabla p, \quad (2a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2b)$$

where we have assumed that the disturbances are inviscid. Our interest is in the case where the perturbations are the instability waves of the base flow; these waves may be considered inviscid when the effective Reynolds number of the base flow (which we now assume to be of a free-shear-flow type, i.e. mixing layers, wakes, jets) is larger than about 500 (Betchov & Szewczyk 1963). Actually, most of our analysis is also applicable to wall-bounded shear layers provided that we look at only their inviscid short-wavelength instabilities (of the secondary type) which are known to occur when the original base velocity profile develops an inflection point due to nonlinearities induced by the primary instability wave. In order to be specific about boundary conditions, we restrict our discussion to free-shear layers, however.

To repeat, our principal objective is to study solutions of (2a, b) when the four-vector  $(\mathbf{u}, p)$  represents an instability wave packet which is propagating on a slightly inhomogeneous (but given) base flow. The inhomogeneity, which will be made precise in the next section, is present in both space and time.

The simplicity, success, and beauty of such a study depend critically on the choice of the dependent variables. Motivated by some classical work on plasma instabilities (Bernstein *et al.* 1958), we introduce  $\alpha(\mathbf{x}, t)$  and  $p(\mathbf{x}, t)$  as our new dependent variables, where

$$\mathbf{u} = \frac{D\alpha}{Dt} - \alpha \cdot \nabla \mathbf{U}. \quad (3)$$

The perturbation equations (2a, b) transform into

$$\frac{D^2\alpha}{Dt^2} = -\nabla p + \alpha \cdot \nabla \mathcal{F}, \quad (4a)$$

$$\nabla \cdot \alpha = 0. \quad (4b)$$

Recall that  $\mathcal{F}$  is the acceleration of the base flow and  $p$  is the perturbation pressure. A physical interpretation for  $\alpha = \alpha(\mathbf{x}, t)$  is this: Consider a fluid particle of fixed

identity that occupies the point  $\mathbf{x}$  (at time  $t$ ) in the unperturbed base flow. The position of this same particle in the perturbed flow (at time  $t$ ) is defined to be  $\mathbf{x} + \boldsymbol{\alpha}$ . In other words, the particle that would have been at  $\mathbf{x}$  in the base flow is actually at a slightly different position (namely,  $\mathbf{x} + \boldsymbol{\alpha}$ ) in the disturbed flow. Crudely speaking,  $\boldsymbol{\alpha}$  represents the ‘relative displacement of a fluid particle’; it is a quantity that can be measured experimentally by so-called ‘imaging methods’. We shall be working with (4a, b) rather than (2a, b).

One significant advantage of (4a, b) is that they are derivable from a bilinear Lagrangian formulation. Because of this, the powerful apparatus of kinematic wave theory can be brought to bear on our problem, and this will be done in §7. Since the tensor  $\nabla\mathcal{F}$  is, in general, non-symmetric, a Lagrangian density, involving only  $(\boldsymbol{\alpha}, p)$ , does not exist. Even if such a Lagrangian did exist, as in the case when  $\mathcal{F} = -\nabla P$ , it would be of limited value for instability waves because these waves possess an exponential growth in addition to an oscillatory behaviour. Therefore, the wave is not periodic in either space or time.

A very useful suggestion of Russell (1986) is the introduction of a bilinear Lagrangian density involving a set of dependent variables,  $\Phi = (\boldsymbol{\alpha}, \boldsymbol{\beta}, p, \pi)$ , in which  $(\boldsymbol{\alpha}, p)$  are our physical variables and  $(\boldsymbol{\beta}, \pi)$  are auxiliary variables used in the construction of the Lagrangian.  $\Phi = (\Phi_i)$  is a vector with eight components ( $i = 1, \dots, 8$ ), however, no boldface is used on  $\Phi$ . The bilinear Lagrangian density is

$$L(\dot{\Phi}, \nabla\Phi, \Phi, \mathbf{x}, t) = \frac{D\boldsymbol{\alpha}}{Dt} \cdot \frac{D\boldsymbol{\beta}}{Dt} + p\nabla \cdot \boldsymbol{\beta} + \pi\nabla \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha} \cdot \nabla\mathcal{F} \cdot \boldsymbol{\beta}, \quad (5a)$$

and the corresponding variational principle is expressible as

$$\delta \int dt \int L dx = 0, \quad (5b)$$

where  $\Phi = \partial\Phi/\partial t$ .

There are eight Euler equations (in the sense of variational calculus) associated with (5b). Four of these are simply (4a, b) for  $(\boldsymbol{\alpha}, p)$ , and the other four are

$$\frac{D^2\boldsymbol{\beta}}{Dt^2} = -\nabla\pi + \nabla\mathcal{F} \cdot \boldsymbol{\beta} \quad (6a)$$

$$\nabla \cdot \boldsymbol{\beta} = 0. \quad (6b)$$

Note that  $(p, \pi)$  serve as Lagrange multipliers for the incompressibility conditions on  $(\boldsymbol{\beta}, \boldsymbol{\alpha})$ , respectively.

The equations for  $(\boldsymbol{\alpha}, p)$  and  $(\boldsymbol{\beta}, \pi)$  are decoupled, and for suitable choices of  $(\boldsymbol{\beta}, \pi)$ , the classical ideas of kinematic wave theory can be readily applied to our problem. Roughly speaking, if  $(\boldsymbol{\alpha}, p)$  represent an unstable wave,  $(\boldsymbol{\beta}, \pi)$  will represent the corresponding stable wave. Note that when  $\nabla\mathcal{F}$  is a symmetric tensor,  $(\boldsymbol{\alpha}, p)$  and  $(\boldsymbol{\beta}, \pi)$  satisfy identical equations. In this case, the Lagrangian density (5a) may be reduced to the usual one if we take  $\boldsymbol{\beta} = \boldsymbol{\alpha}$  and  $\pi = p$ .

In §7, we also need certain elementary results that are best summarized here. Let  $L_\phi$  and  $L_\psi$  be two eight-vectors which are obtained by differentiating the Lagrangian density (5a) with respect to  $\dot{\Phi} = \partial\Phi/\partial t$  and  $\Phi$ , respectively. A straightforward explicit calculation shows that

$$L_\phi = \left[ \frac{D\boldsymbol{\beta}}{Dt}, \frac{D\boldsymbol{\alpha}}{Dt}, 0, 0 \right], \quad (7a)$$

$$L_\psi = [\nabla\mathcal{F} \cdot \boldsymbol{\beta}, \boldsymbol{\alpha} \cdot \nabla\mathcal{F}, \nabla \cdot \boldsymbol{\beta}, \nabla \cdot \boldsymbol{\alpha}]. \quad (7b)$$

Both  $L_\phi$  and  $L_\psi$  are written as row vectors. In addition, let  $L_{\nabla\phi}$  be a three-by-eight tensor whose  $(i, j)$  component is obtained by differentiating  $L$  with respect to  $\partial\Phi_i/\partial x_j$ . Once again, an explicit calculation on (5a) shows that

$$L_{\nabla\phi} = \left[ U \frac{D\beta}{Dt} + \pi I, U \frac{D\alpha}{Dt} + pI, 0, 0 \right], \tag{7c}$$

where  $I$  is the three-by-three idemtensor and the two zeros stand for two three-dimensional column vectors with zero entries.

Finally, we quote a profound result of Hayes (1970) on which our discussion in §7 will be based. Let us assume, for the moment, that the Euler equations, (4a, b) and (6a, b), have solutions which are periodic in a real parameter, say  $\theta$ , of period  $\mu$ .  $\theta$  is the phase-shift parameter of Hayes so that we have

$$\Phi(\mathbf{x}, t; \theta) = \Phi(\mathbf{x}, t; \theta + \mu). \tag{8}$$

An explicit representation of  $\Phi(\mathbf{x}, t; \theta)$  will be constructed in a later section. It suffices here to observe the identity

$$\frac{\partial}{\partial t} (L_\psi \cdot \Phi_\theta) + \nabla \cdot (L_{\nabla\psi} \cdot \Phi_\theta) = \frac{\partial L}{\partial \theta}, \tag{9a}$$

where  $\Phi_\theta = \partial\Phi/\partial\theta$  and, to repeat,  $\Phi$  satisfies the Euler equations

$$\frac{\partial L_\psi}{\partial t} + \nabla \cdot L_{\nabla\psi} - L_\psi = 0 \tag{9b}$$

and (9a) arises from (9b) after ‘dotting’ the latter by  $\Phi_\theta$  and rearranging some of the terms. If we now average (9a) over the phase-shift parameter, we obtain

$$\frac{\partial}{\partial t} (\overline{L_\psi \cdot \Phi_\theta}) + \nabla \cdot (\overline{L_{\nabla\psi} \cdot \Phi_\theta}) = 0, \tag{9c}$$

where the overbar denotes ‘phase averaging’; that is,

$$\overline{(\cdot)} = \frac{1}{\mu} \int_0^\mu (\cdot) d\theta. \tag{9d}$$

Note that, upon averaging, the right-hand side of (9a) vanishes since the Lagrangian density,  $L$ , is periodic in the phase-shift parameter,  $\theta$ .

Equation (9c) is a basic conservation law which will be used, in several different ways, in later sections. Our final results (§§6 and 7) will involve concepts such as frequency, growth rate, wavenumber, and a small parameter  $\epsilon$  (which is a measure of the inhomogeneity), but it is important to realize that the existence of (9c) is completely independent of these concepts (Hayes 1970).

In closing, we note that the perturbation equations for  $(\mathbf{u}, p)$  also possess a Lagrangian in terms of Clebsch potentials (Seliger & Whitham 1968). However, in instability theory, we need a bilinear Lagrangian, and this is especially simple to construct for our new dependent variables  $(\alpha, p)$ .

### 3. Slowly diverging base flows and related ideas

Let us now consider approximately ‘unidirectional’ base flows and write the functional dependence of the base velocity profile,  $U$ , on the space coordinate,  $\mathbf{x}$ , as

$$U = U(y, X), \tag{10a}$$

where  $\mathbf{y}, X$  denote the so-called lateral and longitudinal coordinates (Bretherton 1968); the dependence on time is for the moment suppressed in (10a). The two subspaces  $\mathbf{y}$  and  $X$  are mutually perpendicular, and their direct sum comprises the physical space  $\mathbf{x}$ . We next assume that the velocity profile depends slowly on the longitudinal coordinates in the sense that the dependence on  $X$  occurs through a quantity  $\xi = \epsilon X$ , where  $0 < \epsilon \ll 1$ . The parameter  $\epsilon$  is a measure of the inhomogeneity of the flow in longitudinal space. Finally, we assume that the dependence of  $U$  on time also occurs through a slow variable, say  $\tau = \epsilon t$ . Thus, we write

$$U = U(\mathbf{y}, \xi, \tau) \quad (10b)$$

and note that in (10b), the derivatives of  $U$ , with respect to each of its arguments, are of order unity. Our primary independent variables in the rest of this paper are  $\mathbf{y}$  (lateral space),  $\xi$  (longitudinal space), and  $\tau$  (time).

The separation of physical space into lateral and longitudinal spaces implies that the basic spatial differential operator,  $\nabla$ , becomes

$$\nabla \rightarrow \nabla_{\perp} + \epsilon \nabla, \quad (11a)$$

where  $\nabla_{\perp}$  and  $\nabla$  denote the gradient operators in lateral and longitudinal space, respectively. In the latter, differentiations are with respect to the variable  $\xi$ . (Note that for simplicity of notation,  $\nabla$  is used to denote two different things, namely,  $\partial/\partial \mathbf{x}$  and  $\partial/\partial \xi$ ; *from here on*,  $\nabla = \partial/\partial \xi$ .)

Clearly, (10b) describes a flow which is changing slowly in longitudinal space. As a result, from the mass continuity of the base flow, (1b), the lateral velocity components will be of  $O(\epsilon)$  under the assumption that at  $\mathbf{y} = \infty$ , the lateral velocity is small. Therefore, it is useful to rescale these components, and to write

$$U \rightarrow \epsilon V + U, \quad (11b)$$

where  $V$  and  $U$  denote the velocity in lateral and longitudinal spaces; both of these velocities are of order unity. (Once again, there is a slight ambiguity of notation in the sense that  $U$  stands for two different things; *from now on*,  $U$  will be the velocity in longitudinal space.)

For  $\epsilon = 0$ , the base flow is said to be parallel, and the disturbance equations (4a, b) have solutions of the form

$$\alpha = \hat{\alpha}(\mathbf{y}) \exp(\omega t + i\mathbf{k} \cdot X) + \text{c.c.}, \quad (12a)$$

where  $\omega$  and  $\mathbf{k}$  are the (complex) frequency and (complex) wavenumber vector, respectively. Whenever  $\alpha$  is required to vanish at  $\mathbf{y} = \infty$ , the wavenumber and frequency are connected by a complex dispersion relation,

$$\omega = \omega(\mathbf{k}), \quad (12b)$$

and  $\hat{\alpha}(\mathbf{y})$  is essentially an eigenfunction in lateral space. Therefore, instability waves are modal waves, and the lateral and longitudinal spaces above are identical with the cross-space and propagation space of Hayes (1970). Note that in (12a), c.c. denotes the complex conjugate of all the explicitly written terms to the right of the equal sign,  $i = (-1)^{\frac{1}{2}}$ , and the perturbation pressure is also described by a modal representation in the form of (12a). Furthermore, observe that the imaginary unit,  $i$ , is missing from our definition of the complex frequency,  $\omega$ . (In the terminology of stability theory,  $\omega$  is called the complex growth rate; in order to bridge the gap between stability and kinematic wave theories, we call  $\omega$  the complex frequency,



although it is the imaginary part of  $\omega$  that represents the oscillatory character of the wave.)

Let us now apply decompositions (11 *a, b*) to the base flow equations, (1 *a, b*). After expanding  $\mathcal{F}$  as

$$\mathcal{F} = \mathcal{F}^{(0)} + \epsilon \mathcal{F}^{(1)} + \epsilon^2 \mathcal{F}^{(2)} \tag{13a}$$

and substituting it into (1 *a, b*), we find

at  $O(\epsilon^0)$ : 
$$\mathcal{F}^{(0)} = 0; \tag{13b}$$

at  $O(\epsilon)$ : 
$$\frac{\partial U}{\partial \tau} + (U \cdot \nabla + V \cdot \nabla_{\perp}) U = \mathcal{F}^{(1)}, \tag{13c}$$

$$\nabla \cdot U + \nabla_{\perp} \cdot V = 0; \tag{13d}$$

at  $O(\epsilon^2)$ : 
$$\frac{\partial V}{\partial \tau} + (U \cdot \nabla + V \cdot \nabla_{\perp}) V = \mathcal{F}^{(2)}. \tag{13e}$$

In other words,  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)}(\mathbf{y}, \xi, \tau) = \mathcal{F}_{\parallel}^{(1)}$  is in longitudinal space (indicated by the subscript  $\parallel$ ) and  $\mathcal{F}^{(2)} = \mathcal{F}^{(2)}(\mathbf{y}, \xi, \tau) = \mathcal{F}_{\perp}^{(2)}$  is in lateral space (indicated by the subscript  $\perp$ ). These results are exact, subject to the slowly varying assumption.

If the force  $\mathcal{F}$  is given by the negative of the pressure gradient of the base flow,  $\mathcal{F} = -\nabla P$ , with

$$P = P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)}, \tag{14a}$$

then from (11 *a*) and (13 *a, b*) we find

at  $O(\epsilon^0)$ : 
$$\nabla_{\perp} P^{(0)} = 0 \tag{14b}$$

or  $P^{(0)} = P^{(0)}(\xi, \tau)$ ;

at  $O(\epsilon)$ : 
$$\mathcal{F}_{\parallel}^{(1)} = -\nabla P^{(0)} \tag{14c}$$

and  $P^{(1)}$  is independent of  $\mathbf{y}$ ;

at  $O(\epsilon^2)$ : 
$$\mathcal{F}_{\perp}^{(2)} = -\nabla_{\perp} P^{(2)}(\mathbf{y}, \tau) \tag{14d}$$

and  $P^{(1)}, P^{(2)}$  are independent of  $\xi$ . Thus we find that  $P^{(1)}$  depends only on  $\tau$ , and by the boundary condition at  $\mathbf{y} = \infty$  we set  $P^{(1)} = 0$ .

Let us now see what happens in a laminar flow where, in addition to pressure gradients, viscous forces are also present. If we choose  $\epsilon = O(Re^{-1})$ , where  $Re \gg 1$  is the Reynolds number of the base flow, then we find

$$\mathcal{F}_{\parallel}^{(1)} = \nabla_{\perp}^2 U(\mathbf{y}, \xi, \tau), \tag{15}$$

where  $\nabla_{\perp}^2$  denotes the Laplacian operator in lateral space. If both pressure gradients and viscous forces are present,  $\mathcal{F}_{\parallel}^{(1)}$  is given by the sum of the two terms on the right-hand sides of (14 *c*) and (15). We shall not give the corresponding expression for  $\mathcal{F}_{\perp}^{(2)}$  in the presence of viscosity since, in the rest of this analysis, we ignore terms of  $O(\epsilon^2)$ ; we shall investigate only the lowest-order effect of the inhomogeneity.

At this point, we are in a position to make some detailed comments on the work of Russell (1986). In §3 of his paper, Russell goes through a series of transformations to arrive at his equations (3.10 *a, b*) and (3.11) which, we believe, roughly correspond to our (4 *a, b*) and (3), respectively. The essential term  $\alpha \cdot (\nabla_{\perp} + \epsilon \nabla) \mathcal{F}$  is missing from his equations; Russell assumes from the outset that the base flow is inviscid so that  $\mathcal{F}$  is minus the unperturbed pressure gradient. In this case, the term in question is no greater than  $O(\epsilon^2)$ , and thus it may be ignored in the lowest-order theory (see (14 *c*)). On the other hand, in the physically realistic situation when the spreading of

a free-shear flow is caused by laminar or turbulent stresses,  $\alpha \cdot (\nabla_{\perp} + \epsilon \nabla) \mathcal{F}$  is of  $O(\epsilon)$  (see (15)), and the term in question must be kept, even in the lowest-order theory. In other words, Russell's work sheds no information on the simplest and most relevant of problems, exemplified by the evolution of an instability wave packet in a spreading mixing layer or jet (see the work of Gaster *et al.* 1985).

**4. Disturbance equations – high-frequency ansatz**

Our principal aim in this section is to develop an approximate solution to (4a, b) for small values of the inhomogeneity parameter,  $\epsilon$ . Here, and in the rest of this paper, we take advantage of the lateral-longitudinal space decomposition and explicitly substitute (11a, b) into (4a, b). The relevant equations become somewhat cumbersome but are needed in this form for the application of the high-frequency ansatz.

In a very straightforward way, we obtain for our perturbation momentum and continuity equations

$$\begin{aligned} \epsilon^2 \alpha_{\tau\tau} + 2\epsilon^2 (U \cdot \nabla + V \cdot \nabla_{\perp}) \alpha_{\tau} + \epsilon^2 (UU : \nabla \nabla + 2UV : \nabla_{\perp} \nabla + VV : \nabla_{\perp} \nabla_{\perp}) \alpha \\ = -(\nabla_{\perp} + \epsilon \nabla) p - \mathcal{F} \cdot (\nabla_{\perp} + \epsilon \nabla) \alpha + \alpha \cdot (\nabla_{\perp} + \epsilon \nabla) \mathcal{F} \end{aligned} \tag{16a}$$

$$(\nabla_{\perp} + \epsilon \nabla) \cdot \alpha = 0. \tag{16b}$$

To emphasize,  $\tau = \epsilon t$  is a slow time,  $(\cdot)_{\tau} = \partial(\cdot)/\partial\tau$ ;  $\xi = \epsilon X$  is a slow space scale;  $y$  is the lateral coordinate;  $\nabla_{\perp} = \partial/\partial y$  and  $\nabla = \partial/\partial \xi$  represent the gradient operators in the lateral and longitudinal spaces, respectively; and  $\mathcal{F} = \epsilon \mathcal{F}_{\parallel}^{(1)} + \dots$  is the acceleration of the base flow.  $U(y, \xi, \tau)$  and  $V(y, \xi, \tau)$  are the base flow velocities in the two spaces, and the actual base velocity is given by  $U + \epsilon V$ .

We next separate our disturbances into rapidly growing and oscillating parts and more slowly varying ‘amplitude functions’. In the spirit of the high-frequency ansatz (Lewis 1965), we write

$$\alpha = E \hat{\alpha} + \text{c.c.}, \quad p = E \hat{p} + \text{c.c.} \tag{17a}$$

with 
$$E = \exp(\phi/\epsilon + i\theta), \quad i = (-1)^{\frac{1}{2}}, \tag{17b}$$

where  $\phi = \phi(\xi, \tau)$  is the complex phase,  $\theta$  is the (real) phase-shift parameter, and c.c. denotes the complex conjugate of all the explicitly written terms to the right of the equal sign. Since we shall not be dealing with nonlinear entities until we discuss the Lagrangian formulation, we shall omit c.c. in this section and in §§5 and 6. In addition to (17), the ‘amplitude functions’  $\hat{\alpha}$  and  $\hat{p}$  are expanded as

$$\hat{\alpha} = \alpha^{(0)} + \epsilon \alpha^{(1)} + \dots, \tag{18a}$$

$$\hat{p} = p^{(0)} + \epsilon p^{(1)} + \dots, \tag{18b}$$

where the variables in (18) depend on  $y$ ,  $\xi$ , and  $\tau$ . The characteristic wavelength of a typical instability wave (say, the most unstable wave in the packet) is on the order of the characteristic dimension of a free-shear layer in lateral (cross) space. In our normalization scheme, the wavelength of the individual waves is of  $O(1)$ , and the presence of the factor  $\epsilon^{-1}$  in (17b) implies that this wavelength is much smaller than the lengthscale on which the inhomogeneity of the base flow occurs in longitudinal space.

Based on our high-frequency ansatz, it is possible to show (see (26)) that  $\partial\phi/\partial\tau$  and  $-i\nabla\phi$  are the local frequency and wavenumber. These change by an  $O(1)$  amount on the lengthscale of the inhomogeneity; on the other hand, the change in these

quantities in one wavelength is  $O(\epsilon)$ . It is also possible to consider a more general situation in which the fractional change in the wavenumber per wavelength is some other (i.e. fractional) power of  $\epsilon$  (Nayfeh 1980).

We next substitute (17) and (18) into (16) and collect terms of  $O(\epsilon^0)$  and of  $O(\epsilon)$ . After noting the differentiation rule

$$\epsilon \nabla \alpha = E(\nabla \phi + \epsilon \nabla) \hat{\alpha} \tag{19}$$

and its counterpart for  $\epsilon \partial / \partial \tau$ , we obtain

$$\text{at } O(\epsilon^0): \quad \mathcal{L}(\phi_\tau, -i\nabla\phi)(\alpha^{(0)}, p^{(0)}) = 0, \tag{20a}$$

where  $\phi_\tau = \partial\phi/\partial\tau$  and  $\mathcal{L}(\omega, \mathbf{k})$  is a linear operator whose action on  $(\alpha^{(0)}, p^{(0)})$  is defined by

$$\omega_0^2 \alpha^{(0)} + i\mathbf{k}p^{(0)} + \nabla_\perp p^{(0)} \quad (\text{a 3-vector}), \tag{20b}$$

$$i\mathbf{k} \cdot \alpha^{(0)} + \nabla_\perp \cdot \alpha^{(0)} \quad (\text{a scalar}), \tag{20c}$$

where

$$\omega_0(\omega, \mathbf{k}) = \omega + i\mathbf{k} \cdot \mathbf{U}. \tag{20d}$$

The operator  $\mathcal{L}$  and some of its characteristics will be discussed in the next section. It suffices here to note that  $\omega = \phi_\tau$  is a (complex) frequency and  $\mathbf{k} = -i\nabla\phi$  is a (complex) wavenumber in propagation space. In a similar (complex) sense,  $\omega_0$  is the Doppler-shifted frequency seen by an observer moving with the base velocity  $\mathbf{U}$  in longitudinal space.

As usual, at the next order in  $\epsilon$ , we obtain inhomogeneous equations for  $(\alpha^{(1)}, p^{(1)})$ . These have the form

$$\text{at } O(\epsilon): \quad \mathcal{L}(\phi_\tau, -i\nabla\phi)(\alpha^{(1)}, p^{(1)}) = (R, R_c), \tag{21a}$$

where

$$R = -\frac{D_\parallel \omega_0}{D\tau} \alpha^{(0)} - 2\omega_0 \frac{D_\parallel \alpha^{(0)}}{D\tau} - (V \cdot \nabla_\perp U \cdot \nabla\phi) \alpha^{(0)} - 2\omega_0 V \cdot \nabla_\perp \alpha^{(0)} - \nabla p^{(0)} + \alpha^{(0)} \cdot \nabla_\perp \mathcal{F}_\parallel^{(1)}, \tag{21b}$$

$$R_c = -\nabla \cdot \alpha^{(0)}, \tag{21c}$$

and

$$\frac{D_\parallel}{D\tau} = \frac{\partial}{\partial \tau} + \mathbf{U} \cdot \nabla. \tag{21d}$$

There are several points to note. First, in (21b),  $\omega_0 = \omega_0(\phi_\tau, -i\nabla\phi)$  (i.e.  $\omega$  is replaced by  $\phi_\tau$  and  $\mathbf{k}$  by  $-i\nabla\phi$ , where  $\phi$  is the complex phase); second,  $D_\parallel/D\tau$  denotes a substantial derivative in longitudinal space; and, third, (21a) is to be interpreted in the following way: Replace  $(\alpha^{(0)}, p^{(0)})$  by  $(\alpha^{(1)}, p^{(1)})$  in (20b, c) and then make an equation out of these by placing  $R$  and  $R_c$  on the right-hand sides.

Since the homogeneous equation (20a) has non-trivial solutions with null boundary conditions at  $\mathbf{y} \rightarrow \infty$ , the operator  $\mathcal{L}$  is singular. In this case, the inhomogeneous equation (21a) will have a solution (in fact, infinitely many) if and only if the right-hand side,  $(R, R_c)$ , satisfies the so-called solvability condition. This is discussed in the next section.

### 5. Modes and solvability

The first task ahead of us is to solve the modal version of (20a), namely,

$$\mathcal{L}(\omega, \mathbf{k})(\alpha_m, p_m) = 0, \tag{22}$$

with  $\alpha_m$  and  $p_m$  vanishing at  $\mathbf{y} \rightarrow \infty$ , where the subscript  $m$  denotes 'mode'. It turns out (Betchov & Criminale 1967; Drazin & Reid 1981) that (22) with the specified boundary conditions will have a non-trivial solution if and only if the frequency,  $\omega$ , and wavenumber,  $\mathbf{k}$ , obey the complex dispersion relation,

$$\omega = \omega(\mathbf{k}, \boldsymbol{\xi}, \tau), \quad (23a)$$

which now depends on the slow scales  $\boldsymbol{\xi}$  and  $\tau$  because of the inhomogeneity of the base flow. Since  $\omega$  depends on  $\mathbf{k}$ , the modes have functional forms

$$\alpha_m = \alpha_m(\mathbf{y}, \boldsymbol{\xi}, \tau, \mathbf{k}), \quad p_m = p_m(\mathbf{y}, \boldsymbol{\xi}, \tau, \mathbf{k}) \quad (23b)$$

(i.e.  $\alpha_m$  and  $p_m$  are explicitly independent of  $\omega$ ).

The form of the continuity and momentum equations, (20a), together with expressions (20b, c), suggests that we should explicitly decompose  $\alpha_m$  into its components in the cross- and propagation spaces. In order to accomplish this, we write

$$\alpha_m = \mathbf{Q} - i\mathbf{W}, \quad (24)$$

where  $\mathbf{Q} = \mathbf{Q}(\mathbf{y}, \boldsymbol{\xi}, \tau, \mathbf{k})$  and  $\mathbf{W} = \mathbf{W}(\mathbf{y}, \boldsymbol{\xi}, \tau, \mathbf{k})$  are vectors in the cross- and propagation spaces, respectively. The subscript  $m$  is omitted on  $\mathbf{Q}$  and  $\mathbf{W}$  because it is only in this modal context that we use these two letters.

After substituting (24) into (22) and performing a few more-or-less obvious algebraic manipulations, we arrive at

$$p_m = \omega_0^2 \frac{\mathbf{k} \cdot \mathbf{W}}{\mathbf{k} \cdot \mathbf{k}}, \quad (25a)$$

$$\mathbf{Q} = -\omega_0^{-2} \nabla_{\perp} \left( \frac{\omega_0^2 \mathbf{k} \cdot \mathbf{W}}{\mathbf{k} \cdot \mathbf{k}} \right), \quad (25b)$$

$$\mathbf{W} = \mathbf{k} \frac{\mathbf{k} \cdot \mathbf{W}}{\mathbf{k} \cdot \mathbf{k}}, \quad (25c)$$

with

$$\omega_0 = \omega_0(\omega, \mathbf{k}) = \omega + i\mathbf{k} \cdot \mathbf{U}, \quad (25d)$$

and  $(\mathbf{k} \cdot \mathbf{W})$  is what we call a *scalar mode*, which satisfies a version of the Rayleigh stability equation

$$\mathcal{R}(\mathbf{k} \cdot \mathbf{W}) = \nabla_{\perp} \cdot \{ \omega_0^{-2} \nabla_{\perp} [ \omega_0^2 (\mathbf{k} \cdot \mathbf{W}) ] \} - \mathbf{k} \cdot \mathbf{k} (\mathbf{k} \cdot \mathbf{W}) = 0. \quad (25e)$$

We call  $\mathcal{R}(\cdot)$  the Rayleigh operator.

In actual practice, the dispersion relation, (23a), and the scalar mode,  $(\mathbf{k} \cdot \mathbf{W})$ , come from solving (25e), for a given  $\mathbf{k}$ ,  $\boldsymbol{\xi}$ , and  $\tau$ , with boundary condition  $\mathbf{k} \cdot \mathbf{W} \rightarrow 0$  as  $\mathbf{y} \rightarrow \infty$ . Typically, these solutions can be obtained only numerically. Once the scalar mode is calculated, the physical quantities of interest,  $(\alpha_m, p_m)$ , may be obtained from (25a, b, c) and (24). The normalization of the modes is irrelevant.

The simplicity and generality of the previous decomposition is another compelling reason for using  $(\boldsymbol{\alpha}, p)$  in place of  $(\mathbf{u}, p)$ . For the latter variables, the components of  $\mathbf{u}$  in the cross- and propagation spaces are not decoupled because of the presence of the base flow velocity gradient in (2a). We observe from (25c) that the component of the displacement,  $\alpha_m$ , in propagation space is along the wavenumber,  $\mathbf{k}$ , and is proportional to the perturbation pressure  $p_m$ . Similar results hold for the perturbation *velocity* field of an acoustic wave (Hayes 1968). Finally, note that  $\omega_0$  also depends on the slow variables  $\boldsymbol{\xi}$  and  $\tau$  and on  $\mathbf{y}$ , but this dependence is not indicated explicitly in (20d) or (25d).

It is now clear from (22), (20a), and (23a) that the lowest-order equations for  $(\alpha^{(0)}, p^{(0)})$  will have a solution if and only if

$$\phi_\tau = \omega(-i\nabla\phi, \xi, \tau). \quad (26)$$

We interpret (26) as a first-order partial differential equation for the complex phase  $\phi$ . Its solution by the method of (complex) characteristics will lead to (complex) rays (Gaster 1981; Itoh 1981; Russell 1986). We shall not discuss these rays at this point because our primary interest is in the 'transport' or amplitude equations. Of course, (26) need not be solved by the method of characteristics, so rays (complex or otherwise) are not essential to our theory.

The lowest-order solutions must be proportional to  $(\alpha_m, p_m)$ ; to express this, we write

$$\alpha^{(0)} = A\alpha_m, \quad p^{(0)} = Ap_m, \quad (27)$$

where  $A = A(\xi, \tau)$  is a slowly varying complex amplitude (so far arbitrary), and  $(\alpha_m, p_m)$  on the right-hand sides of (27) are evaluated at  $\mathbf{k} = -i\nabla\phi$ . The partial differential equation for the amplitude,  $A$ , is determined from the next-order solution for  $(\alpha^{(1)}, p^{(1)})$  by enforcing the solvability condition on the inhomogeneous terms  $(\mathbf{R}, R_c)$  (see (21a)).

The second task ahead of us is to accomplish this explicitly and simply. Define an inner product of two scalar complex-valued functions, say  $f$  and  $g$ , by the usual formula

$$(f, g) = \int f\tilde{g} \, dy, \quad (28)$$

where the integration is taken over the cross-space and the tilde denotes complex conjugation. In order to enforce solvability, it is very convenient to transform (21a) into a single equation for the quantity  $(\nabla\phi \cdot \alpha^{(1)})$  – the steps for this operation are exactly the same as those leading to (25e). The final result is

$$\mathcal{R}(-i\nabla\phi \cdot \alpha^{(1)}) = (\nabla\phi)^2 R_c - \nabla_\perp \cdot \frac{\mathbf{R}_\perp (\nabla\phi)^2}{\omega_0^2} + i\nabla_\perp \cdot [\omega_0^{-2} \nabla_\perp (\mathbf{R}_\parallel \cdot \nabla\phi)], \quad (29a)$$

where, in (29a), we have separated the vector  $\mathbf{R}$  of (21b) into its components in cross-space,  $\mathbf{R}_\perp$ , and in propagation space,  $\mathbf{R}_\parallel$ , via

$$\mathbf{R} = \mathbf{R}_\perp + i\mathbf{R}_\parallel. \quad (29b)$$

Note the presence of  $i$  in (29b) and the facts that in (29a)  $\omega_0$  is evaluated at  $\omega = \phi_\tau$ , and the wavenumber  $\mathbf{k}$  in the Rayleigh operator and in  $\omega_0$  is replaced by  $-i\nabla\phi$ .

The precise statement of solvability is that the right-hand side of (29a) must be orthogonal to solutions of the adjoint Rayleigh equation. Two functions are orthogonal if their inner product vanishes. Under our definition of orthogonality, (28), the adjoint Rayleigh equation for a variable, say  $h$ , and the corresponding solution are

$$\mathcal{R}^*(h) = \tilde{\omega}_0^2 \nabla_\perp \cdot (\tilde{\omega}_0^{-2} \nabla_\perp h) - \tilde{\mathbf{k}} \cdot \tilde{\mathbf{k}} h \quad (30a)$$

and

$$h = \tilde{\omega}_0^2 \tilde{\mathbf{k}} \cdot \tilde{\mathbf{W}}, \quad (30b)$$

where, to repeat,  $\mathcal{R}^*$  is the adjoint operator, the tilde denotes complex conjugation, and the adjoint solution,  $h$ , is directly expressible in terms of  $\mathbf{W}$  (which is essentially the component of the displacement,  $\alpha_m$ , in propagation space). We require the adjoint solution to vanish as  $y \rightarrow \infty$ ; this is automatically satisfied because  $\mathbf{k} \cdot \mathbf{W}$  vanishes.

In order to enforce the solvability condition (or orthogonality), we multiply the right-hand side of (29a) with the conjugate of  $h$  and integrate over cross-space. The

cross-space gradient operator,  $\nabla_{\perp}$ , is removed from the integrand by successive integrations by parts and by invoking Rayleigh's equation, (25e), at suitable points in the analysis. The final result is remarkably simple:

$$\int (\mathbf{R}_{\perp} \cdot \mathbf{Q} - \mathbf{R}_{\parallel} \cdot \mathbf{W} - R_c p_m) dy = 0, \quad (31)$$

where the integral in (31) is taken over cross-space and  $\mathbf{R}_{\perp}$  and  $\mathbf{R}_{\parallel}$  are defined by (29b). Equation (31) provides a single (complex) constraint which is used to determine the (complex) amplitude  $A(\xi, \tau)$ . This is done in the next section.

## 6. Amplitude equations and group velocity

In order to enforce solvability condition, (31), explicitly, we use (21b, c) for  $\mathbf{R}$  and  $R_c$ , (27) for  $(\alpha^{(0)}, p^{(0)})$ , and (24) and (25) for  $(\alpha_m, p_m)$ . The algebra is fairly lengthy but moderately straightforward, and we simply quote the final result followed by a short description of the actual manipulations. Let us define two complex quantities

$$\mathcal{A} = A^2 \int \omega_0 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) dy, \quad (32a)$$

$$\mathcal{A}_f = A^2 \int U \omega_0 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) dy, \quad (32b)$$

in which the integrals are known once the modes are calculated. (The subscript  $f$  stands for flux.) In other words, the only unknown in  $\mathcal{A}$  is the complex amplitude  $A$ , and the dependence of the former on the latter is quadratic. Similar remarks hold for  $\mathcal{A}_f$ . One of the key results of this paper is our basic amplitude equation (in its first version)

$$\frac{\partial \mathcal{A}}{\partial \tau} + \nabla \cdot \left[ \mathcal{A}_f + iA^2 \int \mathbf{W} p_m dy \right] - iA^2 \int \mathbf{Q} \cdot \nabla_{\perp} \mathcal{F}_{\parallel}^{(1)} \cdot \mathbf{W} dy = 0, \quad (32c)$$

in which the coefficients of  $A^2$  are also known from modal calculations and the base flow. We shall give a detailed physical interpretation for (32c), but only after we present a second version for it.

The first and last terms in (32c) arise from the  $\tau$ -partial derivatives in and the last term of (21b), respectively. The second term under the divergence comes from the pressure gradient in (21b) and the term,  $R_c = -\nabla \cdot \alpha^{(0)}$ , in (21c). Finally, the remaining terms in (21b) yield the first term under the divergence in (32c) plus one extra term which is a pure divergence in *cross-space*. Upon integration over cross-space, only the term  $\mathcal{A}_f$  survives.

We now briefly show that the argument of the divergence in (32c) may be rewritten in terms of the (complex) group velocity. In order to do this, we differentiate the modal equations, (22), with respect to the wavenumber,  $\mathbf{k}$ , and dot the resultant equations from the left with an arbitrary vector, say  $\mathbf{K}$ , in propagation space. The first significant intermediate result is

$$\mathcal{L}(\omega, \mathbf{k}) \left( \mathbf{K} \cdot \frac{\partial \alpha_m}{\partial \mathbf{k}}, \mathbf{K} \cdot \frac{\partial p_m}{\partial \mathbf{k}} \right) = (\mathbf{R}', R'_c), \quad (33a)$$

where 
$$R' = -2\omega_0 \mathbf{K} \cdot \frac{\partial \omega_0}{\partial \mathbf{k}} \boldsymbol{\alpha}_m - i\mathbf{K} p_m, \tag{33b}$$

$$R'_c = -i\mathbf{K} \cdot \boldsymbol{\alpha}_m, \tag{33c}$$

$$\mathbf{G} = i \frac{\partial \omega}{\partial \mathbf{k}} \quad (\xi, \tau \text{ constant}). \tag{33d}$$

Clearly,  $\mathbf{G}$  is the group velocity. Since (33a) must have a solution, the solvability condition (31) must be satisfied for an arbitrary  $\mathbf{K}$ . This implies (after a bit of algebra)

$$\int [U\omega_0(\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) + iW p_m] dy = \mathbf{G} \int \omega_0(\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) dy, \tag{34}$$

which is called the basic identity by Lewis (1965) in the special case of local hyperbolic waves.

We may now eliminate the argument of the divergence in (32c) in favour of the group velocity via (34). The second version of our amplitude equation thus becomes

$$\frac{\partial \mathcal{A}}{\partial \tau} + \nabla \cdot (\mathbf{G}\mathcal{A}) - iA^2 \int \mathbf{Q} \cdot \nabla_{\perp} \mathcal{F}_{\parallel}^{(1)} \cdot \mathbf{W} dy = 0. \tag{35}$$

Equation (35) is in the form of a standard conservation law. Volume integrals of  $\mathcal{A}$  are conserved in propagation space subject to the fluxes ( $\mathbf{G}\mathcal{A}$ ) and the source term

$$\sigma = iA^2 \int \mathbf{Q} \cdot \nabla_{\perp} \mathcal{F}_{\parallel}^{(1)} \cdot \mathbf{W} dy. \tag{36}$$

Roughly speaking, the source term is proportional to the cross-space integral (i.e. an ‘average’) of the changes in the base flow accelerations in cross-space projected onto the propagation space. For perfectly parallel and steady base flows,

$$\mathcal{F} = \frac{DU}{Dt} \equiv 0, \tag{37}$$

and in this case  $\mathcal{A}$  obeys a pure conservation law (without a source term). Actually, much more can be said for parallel flows since, in this case, the wavenumber,  $\mathbf{k}$ , satisfies

$$\frac{\partial \mathbf{k}}{\partial \tau} + \mathbf{G} \cdot \nabla \mathbf{k} = 0 \tag{38a}$$

and (37), (38a), and (35) imply (see Itoh 1980)

$$\frac{\partial A^2}{\partial \tau} + \nabla \cdot (\mathbf{G}A^2) = 0. \tag{38b}$$

This special result can also be derived by the saddle-point method, which shows that the group velocity is real and is given by  $\xi/\tau$  (Gaster 1975). For inhomogeneous base flows, the group velocity is complex in general.

### 7. Lagrangian considerations

We have now reached the centre of gravity of this paper, and the reader who is only interested in using the amplitude equations (32c) or (35) can proceed as follows: He can calculate the frequency,  $\omega$ , and the pressure mode,  $p_m$ , for a given base flow,

as a function of  $\mathbf{k}$ ,  $\xi$ , and  $\tau$ , using his favourite Rayleigh solver, and then compute  $(\mathbf{k} \cdot \mathbf{W})$  from (25a). Once this quantity is known,  $\mathbf{Q}$  and  $\mathbf{W}$  follow from (25b, c). At this point, essentially all the coefficients of the amplitude equations are known, and the latter may be integrated by (say) numerical methods for partial differential equations in real  $(\xi, \tau)$ -space or by methods for ordinary differential equations in complex  $(\xi, \tau)$ -space, along the rays. In a separate study, we are comparing the results from these two approaches. We remind the reader that the acceleration of the base flow is written as  $\mathcal{F} = \epsilon \mathcal{F}^{(1)} + \dots$  so that, in this paper,  $\mathcal{F}^{(1)}$  is considered to be a given quantity. If the amplitude equation is solved as a partial differential equation, (32c) is probably preferable to (35) since the former does not require the calculation of the group velocity.

It is, however, worthwhile to carry the analysis a good bit further using the Lagrangian formalism. In the following three separate subsections, we shall look at three different Lagrangians; we begin with the bilinear Lagrangian of Russell (1986).

### 7.1. Bilinear Lagrangian

The eight-vector  $\Phi$  of the bilinear Lagrangian has components  $(\alpha, \beta, p, \pi)$ ; see (5a). The high-frequency ansatz for  $(\alpha, p)$  is given by (17) and the corresponding expressions for  $(\beta, \pi)$  are

$$\beta = E^{-1} \hat{\beta} + \text{c.c.}, \quad \pi = E^{-1} \hat{\pi} + \text{c.c.}, \quad (39a)$$

where

$$E^{-1} = \exp(-\phi/\epsilon - i\theta). \quad (39b)$$

The boundary condition requires  $(\beta, \pi) \rightarrow 0$  as  $\mathbf{y} \rightarrow \infty$ . It is clear that if  $(\alpha, p)$  represent unstable waves (for which  $\phi_{\text{R}} = \text{Re}(\phi) > 0$ ; the subscript R denotes the real part of a complex quantity), then  $(\beta, \pi)$  stand for stable waves. Also, the ‘amplitude functions’  $(\hat{\beta}, \hat{\pi})$  are expanded as

$$\hat{\beta} = \beta^{(0)} + \epsilon \beta^{(1)} + \dots, \quad \hat{\pi} = \pi^{(0)} + \epsilon \pi^{(1)} + \dots, \quad (39c)$$

where terms of  $O(\epsilon^2)$  and smaller are ignored.

Now, let us note from (4a, b) and (6a, b) that  $(\alpha, p)$  and  $(\beta, \pi)$  obey similar (but not identical) equations so that from the analogue of (20a), we conclude that  $(\beta^{(0)}, \pi^{(0)})$  must be a mode with frequency  $(-\phi_*)$  and wavenumber  $(+i\nabla\phi)$ . In other words, we may obtain  $(\beta^{(0)}, \pi^{(0)})$  from (25) by making the substitution  $\mathbf{k} \rightarrow -\mathbf{k}$  and  $\omega \rightarrow -\omega$ . After observing from (25e) that  $(\mathbf{k} \cdot \mathbf{W})$  is symmetric under this transformation, we conclude that so are  $p_{\text{m}}$  and  $\mathbf{Q}$ , but  $\mathbf{W}$  is antisymmetric. Therefore,

$$\beta^{(0)} = B(\mathbf{Q} + i\mathbf{W}), \quad \pi^{(0)} = Bp_{\text{m}}, \quad (40)$$

where  $B = B(\xi, \tau)$  is a slowly varying amplitude associated with the auxiliary variables  $(\beta, \pi)$ . In general,  $B \neq A$  since at  $O(\epsilon)$ , the gradient of  $\mathcal{F}$  enters differently for  $(\beta, \pi)$  than for  $(\alpha, p)$  (see (4), (6), and (21b)). We shall elaborate on this point at the end of this subsection.

The various terms in (9c) may be evaluated with the help of (7a) and (7c). We tentatively write

$$\overline{L_{\phi} \cdot \Phi_{\theta}} = \Gamma^{(0)} + \epsilon \Gamma^{(1)} + O(\epsilon^2) \quad (41a)$$

and, with this temporary notation, the argument of the divergence in (9c) turns out to be (of course, recall (11b))

$$\overline{L_{\nabla\phi} \cdot \Phi_{\theta}} = U(\Gamma^{(0)} + \epsilon \Gamma^{(1)}) + \epsilon V\Gamma^{(0)} + \mathbf{E}^{(0)} + \epsilon \mathbf{E}^{(1)} + O(\epsilon^2), \quad (41b)$$



where the  $\Gamma$  and  $\mathbf{E}$  in (41) will be defined below (see (43) and (44)). First, however, we determine which of these four quantities we actually need for a conservation law in propagation space if we neglect terms of  $O(\epsilon^2)$  and smaller.

The answer to this question comes from (9c), after noting that  $\partial/\partial t = \epsilon \partial/\partial \tau$  and using (11a) for the divergence operator in (9c). The final result is

$$\epsilon \frac{\partial \Gamma^{(0)}}{\partial \tau} + \nabla_{\perp} \cdot [U\Gamma^{(0)} + \epsilon U\Gamma^{(1)} + \epsilon V\Gamma^{(0)} + \mathbf{E}^{(0)} + \epsilon \mathbf{E}^{(1)}] + \epsilon \nabla \cdot [U\Gamma^{(0)} + \mathbf{E}^{(0)}] + O(\epsilon^2) = 0, \tag{42a}$$

and (42a) implies

$$\text{at } O(\epsilon^0): \quad \nabla_{\perp} \cdot \mathbf{E}^{(0)} = 0; \tag{42b}$$

$$\text{at } O(\epsilon): \quad \frac{\partial \Gamma^{(0)}}{\partial \tau} + \nabla_{\perp} \cdot (V\Gamma^{(0)} + \mathbf{E}^{(1)}) + \nabla \cdot (U\Gamma^{(0)} + \mathbf{E}^{(0)}) = 0. \tag{42c}$$

Note that terms of the type  $\nabla_{\perp} \cdot (U\Gamma^{(0)})$  are absent in the above equations since  $\nabla_{\perp}$  is an operator in cross-space and  $U$  is in propagation space; these two spaces are orthogonal.

The first of these absolute laws, (42b), requires that  $\mathbf{E}^{(0)}$  be solenoidal in cross-space. A direct calculation shows that

$$\mathbf{E}^{(0)} + \epsilon \mathbf{E}^{(1)} + \dots \equiv \pi \alpha_{\theta} + \overline{p \beta_{\theta}}, \tag{43a}$$

$$\text{so that} \quad \mathbf{E}^{(0)} = i\pi^{(0)} \alpha^{(0)} - ip^{(0)} \beta^{(0)} + \text{c.c.} = 2AB W p_m + \text{c.c.} \tag{43b}$$

via (39), (40), (27), and (24). We have also used the fact that  $\Phi = (\alpha, \beta, p, \pi)$  is periodic in the phase-shift parameter  $\theta$  (of period  $\mu = 2\pi$ ). The solenoidal condition is trivially satisfied since  $\mathbf{E}^{(0)}$  lies in propagation space.

From (41a) and (7a), we find

$$\Gamma^{(0)} + \epsilon \Gamma^{(1)} + \dots \equiv \overline{\frac{D\beta}{Dt} \cdot \alpha_{\theta}} + \overline{\frac{D\alpha}{Dt} \cdot \beta_{\theta}} \tag{44a}$$

and a direct evaluation of the right-hand side of (44a) gives

$$\Gamma^{(0)} = -2i \left( \frac{\partial \phi}{\partial \tau} + \mathbf{U} \cdot \nabla \phi \right) \alpha^{(0)} \cdot \beta^{(0)} + \text{c.c.} \tag{44b}$$

$$= -2i AB \omega_0 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) + \text{c.c.}, \tag{44c}$$

where we have used steps similar to those in the derivation of (43b). We shall not give the expressions for  $\Gamma^{(1)}$  and  $\mathbf{E}^{(1)}$  because they do not enter into our conservation law in propagation space.

Observe the extremely important point that (42b, c) are conservation laws in physical space, whereas (32c) and (35) are conservation laws in propagation space only. In the terminology of Hayes (1970), (42b, c) are absolute conservation laws – the corresponding laws in propagation space follow from integration over cross-space. The second absolute law, (42c), tries to correct for the fact that  $\mathbf{E}^{(0)} + \epsilon \mathbf{E}^{(1)} + \dots$  is not exactly solenoidal; this quantity is only solenoidal in an asymptotic sense as  $\epsilon \rightarrow 0$ . Equation (42c) is somewhat inconvenient to use because it involves not only the lowest-order solutions (denoted by the superscript 0) but also the first-order solutions (denoted by the superscript 1). However, the integral of (42c) over cross-space yields

a conservation law in propagation space which involves only the lowest-order solutions. Finally, with the definitions,

$$\mathcal{D} = AB \int \omega_0 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) dy, \quad (45a)$$

$$\mathcal{D}_t = AB \int U \omega_0 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) dy, \quad (45b)$$

the bilinear conservation law in propagation space becomes

$$i \left[ \frac{\partial \mathcal{D}}{\partial \tau} + \nabla \cdot \left( \mathcal{D}_t + iAB \int \mathbf{W} p_m dy \right) \right] + \text{c.c.} = 0. \quad (45c)$$

Equation (45c) is an explicit version of the conservation principle given by Russell (1986) in terms of the usual derivatives (in the sense of kinematic wave theory) of the relevant Lagrangian. Since the group velocity satisfies  $\mathbf{G} = i\partial\omega/\partial\mathbf{k} = i[\partial(-\omega)/\partial(-\mathbf{k})]$ , the argument of the divergence may be replaced by  $(\mathbf{G}\mathcal{D})$  in (45c).

It is now quite interesting to compare amplitude equations (32c) and (45c). They both have the same form, except that (32c) has an all-important source term associated with it. This clearly points out the limitation of the bilinear formalism; although the (bilinear) wave action  $\mathcal{D}$  is conserved, a quantity of far more interest, namely  $\mathcal{A}$  (the ordinary wave action), is not. One cannot conclude anything about the amplitude  $A$  from a knowledge of  $\mathcal{D}$  unless, of course, either  $A$  or  $B$  is computed directly. This takes us back to (32c). From fundamental identity (9c), it is abundantly clear that we can always find an (absolute) conservation law as long as there exist a Lagrangian associated with a problem and a periodic one-parameter family of solutions. Since we may always introduce auxiliary variables, say  $(\boldsymbol{\beta}, \pi), \dots$ , etc., in the construction of the (pseudo) Lagrangian, it is probably not too difficult to find a conserved quantity, analogous to  $\mathcal{D}$ , for any reasonable physical system. The principal difficulty arises when we try to extract something about the amplitude of the actual physical problem from a knowledge of this conserved quantity associated with the (pseudo) Lagrangian.

Since the bilinear wave action,  $\mathcal{D}$ , is conserved but the ordinary one,  $\mathcal{A}$ , is not, the amplitude,  $B = B(\boldsymbol{\xi}, \tau)$ , of the auxiliary variables  $(\boldsymbol{\beta}, \pi)$  must be such that it compensates for changes in  $A(\boldsymbol{\xi}, \tau)$ . In fact, from the transformation  $\omega \rightarrow -\omega, \mathbf{k} \rightarrow -\mathbf{k}$ , equations (32), (4), and (6) and solvability condition (31), it is possible to show that the ordinary wave action  $\mathcal{B}$ , associated with our auxiliary variables  $(\boldsymbol{\beta}, \pi)$ , satisfies

$$\frac{\partial \mathcal{B}}{\partial \tau} + \nabla \cdot \left[ \mathcal{B}_t + iB^2 \int \mathbf{W} p_m dy \right] + iB^2 \int \mathbf{Q} \cdot \nabla_{\perp} \mathcal{F}_{\parallel}^{(1)} \cdot \mathbf{W} dy = 0, \quad (46)$$

where  $\mathcal{B}$  and  $\mathcal{B}_t$  are defined by (32a, b) with  $A$  replaced by  $B$ . It is now clear that a suitable linear combination of (32c) and (46), after a slight rearrangement, results in (45c). The source term  $\sigma$  has opposite effects on amplitudes  $A$  and  $B$  so that the bilinear wave action  $\mathcal{D} \sim AB$  is actually conserved. Note that it is possible to arrive at (45c) using the high-frequency ansatz and without any reference to a Lagrangian, although the existence of one and the fundamental identity of Hayes (1970) make the analysis quite straightforward and provide an independent check.

## 7.2. The classical Lagrangian

If the gradient of the base flow acceleration is a symmetric tensor,  $(\boldsymbol{\alpha}, p)$  and  $(\boldsymbol{\beta}, \pi)$  obey identical equations (see (4) and (6)), and we may choose  $(\boldsymbol{\alpha}, p) = (\boldsymbol{\beta}, \pi)$ . In this

case, the bilinear Lagrangian density, (5a), reduces to the classical one (we ignore an irrelevant factor of 2). It is interesting to inquire into the meaning of the basic identity, (9c), in this case.

The analysis is very similar to the one in the previous subsection for the bilinear Lagrangian; therefore, we shall be extremely brief. For ease of notation, we tentatively write

$$\overline{L_\phi \cdot \Phi_\theta} = [\Gamma^{(0)} + \epsilon \Gamma^{(1)} + O(\epsilon^2)] \exp(2\phi_R/\epsilon), \quad (47a)$$

and with this notation the term under the divergence in (9c) may be expressed as

$$\overline{L_{\nabla\phi} \cdot \Phi_\theta} = [U\Gamma^{(0)} + \epsilon U\Gamma^{(1)} + \epsilon V\Gamma^{(0)} + \mathbf{E}^{(0)} + \epsilon \mathbf{E}^{(1)} + O(\epsilon^2)] \exp(2\phi_R/\epsilon), \quad (47b)$$

where we have used (11b) in deriving (47b); the  $\Gamma$  and  $\mathbf{E}$  remain to be defined (see (48b, c)). Note the formal similarity between (47) and (41); however, in the former, the growth rate  $\phi_R = \text{Re}(\phi) > 0$  appears in the exponential term because  $(\alpha, p)$  represent an unstable wave. Also, the detailed expressions for the  $\Gamma$  and  $\mathbf{E}$  differ from (44) and (43).

The time and space derivatives of (47a, b) may be evaluated after observing (11a), differentiation rule (19), and its counterpart for  $\partial/\partial t = \epsilon \partial/\partial \tau$ . We then find that, at  $O(\epsilon^0)$ , (9c) yields

$$\frac{\partial \phi_R}{\partial \tau} \Gamma^{(0)} + (U\Gamma^{(0)} + \mathbf{E}^{(0)}) \cdot \nabla \phi_R + \frac{1}{2} \nabla_\perp \cdot \mathbf{E}^{(0)} = 0. \quad (48a)$$

On the other hand, at  $O(\epsilon)$ , the analogue of (48a) involves both the lowest- and first-order solutions. Therefore, the usefulness of the  $O(\epsilon)$  equation is somewhat limited and we shall not give it. From the definitions of  $\Gamma^{(0)}$  and  $\mathbf{E}^{(0)}$ , we have

$$\Gamma^{(0)} = -i|A|^2 \omega_0 (\mathbf{Q} \cdot \tilde{\mathbf{Q}} + \mathbf{W} \cdot \tilde{\mathbf{W}}) + \text{c.c.} \quad (48b)$$

and 
$$\mathbf{E}^{(0)} = -i|A|^2 p_m (\tilde{\mathbf{Q}} + i\tilde{\mathbf{W}}) + \text{c.c.} \quad (48c)$$

Finally, the integral of (48a) over cross-space yields the following equation in propagation space:

$$i \left[ \frac{\partial \phi_R}{\partial \tau} \Omega + \left( \Omega_t + i \int \tilde{\mathbf{W}} p_m dy \right) \cdot \nabla \phi_R \right] + \text{c.c.} = 0, \quad (49a)$$

where c.c. denotes the complex conjugate of the first term on the left-hand side,  $|A|^2 = A\tilde{A}$ ,

$$\Omega = \int \omega_0 (\mathbf{Q} \cdot \tilde{\mathbf{Q}} + \mathbf{W} \cdot \tilde{\mathbf{W}}) dy \quad (49b)$$

and 
$$\Omega_t = \int U \omega_0 (\mathbf{Q} \cdot \tilde{\mathbf{Q}} + \mathbf{W} \cdot \tilde{\mathbf{W}}) dy. \quad (49c)$$

The term  $\tilde{\mathbf{Q}}$  is absent from (49a) because  $\tilde{\mathbf{Q}} \cdot \nabla \phi_R = 0$ .

We interpret (49a) as a conservation equation for the growth rate along certain curves in propagation space. This equation can provide a useful check on the numerical accuracy of the solution, although (49a) is not an amplitude equation in the usual sense. As pointed out by Russell (1986), the usefulness of the classical Lagrangian density is quite limited and the bilinear one is far superior for instability waves. This is because (for the classical Lagrangian) the amplitude is not determined until  $O(\epsilon)$  and, at that point, the equation involves the higher-order entities  $\Gamma^{(1)}$  and  $\mathbf{E}^{(1)}$ . There is really no conceptual difficulty in dealing with the term  $\exp(2\phi_R/\epsilon)$  (see (47)), and on this point we disagree with Russell's remarks.

7.3. *The intrinsic Lagrangian*

At lowest order, our governing equations are the modal equations (22). These equations are linear and, for a linear problem, the Lagrangian density is quadratic in the perturbations. On the other hand, solvability condition (31) is essentially a quadratic constraint, and the question is whether we can use it to define a (complex) Lagrangian density relevant to our problem.

In order to do this, we proceed in a way that appears unnatural at first. Let us write the modal equations

$$\omega_0^2 \boldsymbol{\alpha}_m + i\mathbf{k} p_m + \nabla_{\perp} p_m = \mathbf{R}'', \quad (50a)$$

$$i\mathbf{k} \cdot \boldsymbol{\alpha}_m + \nabla_{\perp} \cdot \boldsymbol{\alpha}_m = R''_c \quad (50b)$$

and separate  $\boldsymbol{\alpha}_m$  and  $\mathbf{R}''$  into their components in the cross- and propagation spaces via (24) and (29b). Of course,  $\mathbf{R}'' = R''_c = 0$ , but we shall ignore this fact for a moment; this is the unnatural part of our derivation. With these definitions, the left-hand side of solvability condition (31) becomes

$$\int [\omega_0^2 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) - 2p_m \nabla_{\perp} \cdot \mathbf{Q} - 2p_m \mathbf{k} \cdot \mathbf{W}] dy. \quad (51)$$

If we now say that  $(\boldsymbol{\alpha}_m, p_m)$  is indeed a mode (so that the dispersion relation,  $\omega = \omega(\mathbf{k})$ , is satisfied) then, of course, (51) vanishes. On the other hand, the integrand of (51) exists for *any*  $\mathbf{Q}$ ,  $\mathbf{W}$ ,  $p_m$ ,  $\omega$ , and  $\mathbf{k}$ , and we use it to define the intrinsic Lagrangian,  $A$ , via

$$A = A(\mathbf{Q}, \mathbf{W}, p; \omega, \mathbf{k}, A) = \frac{1}{2} A^2 [(\omega + i\mathbf{k} \cdot \mathbf{U})^2 (\mathbf{Q} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W}) - 2p \nabla_{\perp} \cdot \mathbf{Q} - 2p \mathbf{k} \cdot \mathbf{W}]. \quad (52a)$$

The fact that  $A$  is proportional to  $A^2$  is typical of all Lagrangians associated with a linear problem. It is important to point out that although (52a) was derived from the modal equations, once we have (52a), the complex Lagrangian  $A$  is to be regarded (as usual) as a function of independent variables  $\mathbf{Q}$ ,  $\mathbf{W}$ ,  $p$ ,  $\omega$ ,  $\mathbf{k}$ , and  $A$ .

The variational principle,

$$\delta \int d\tau \int d\xi \int A dy = 0, \quad (52b)$$

in which  $\mathbf{Q}$ ,  $\mathbf{W}$ , and  $p$  are independently varied ( $\omega$ ,  $\mathbf{k}$ , and  $A$  are parameters), gives us the modal equations. For example, the variation  $\delta \mathbf{W}$  yields

$$\omega_0^2 \mathbf{W} - \mathbf{k} p = 0, \quad (52c)$$

which is simply the component of (50a) in propagation space (recall  $\mathbf{R}'' = R''_c = 0$ ). When the modal equations are solved with decaying boundary conditions at  $\mathbf{y} \rightarrow \infty$ , we recover the dispersion relation,  $\omega = \omega(\mathbf{k})$ , and  $A = 0$ .

Clearly, by allowing  $\omega$  and  $\mathbf{k}$  to be independent variables in our intrinsic Lagrangian density, we can define  $A$  even when the dispersion relation is *not* satisfied. This, of course, is needed if we wish to take true partial derivatives with respect to these variables; the extension of  $A$  to non-solutions may be carried out in a number of different ways. The final result is insensitive to the particular choice of non-solutions used (Bretherton 1968; Hayes 1970).

Now, let  $A_{\omega}$  and  $A_{\mathbf{k}}$  denote the usual partial derivatives of the function  $A$  with

respect to two of its independent variables, namely,  $\omega$  and  $\mathbf{k}$ . It is a straightforward matter to show that our amplitude equation, (32c), may be written in the form

$$\int \left[ i \frac{\partial A_\omega}{\partial \tau} + \nabla \cdot A_k + A^2 Q \cdot \nabla_\perp \mathcal{F}_\parallel^{(1)} \cdot \mathbf{W} \right] dy = 0. \tag{53}$$

Equation (53) expresses our result in a form which closely resembles that of kinematic wave theory. (The appearance of  $i$  in front of the first term stems from the fact that  $\omega$  is really the complex growth rate whose imaginary part is the (real) frequency.) The ‘adiabatic invariant’ is the cross-space integral of  $A_\omega$ , the appropriate flux is  $A_k$  (integrated over cross-space), and the source term arises from the variations of the base flow acceleration (integrated over cross-space). The wave action  $\mathcal{A}$  is given by

$$\mathcal{A} = \int A_\omega dy \tag{54a}$$

and the ratio 
$$\int A_k dy / i \int A_\omega dy \tag{54b}$$

is the group velocity,  $\mathbf{G}$ , once we substitute the actual modal solutions into (54a, b).

The most interesting feature of (53) is that, for the special case considered by Russell (1986) (i.e.  $\nabla_\perp \mathcal{F}_\parallel^{(1)} = 0$ ), it expresses a pure conservation principle in the classical form of the Lagrangian formalism of Whitham (1974). In other words, in this case, one does not need to introduce a bilinear Lagrangian at all; the complex Lagrangian density (i.e. the intrinsic Lagrangian) arising from the modal equations and the solvability condition contains the modes and amplitude equation (53). The latter is obtained by setting 
$$\omega = \phi_\tau, \quad \mathbf{k} = -i \nabla \phi \tag{55}$$

in the intrinsic Lagrangian and then taking variations in the complex phase,  $\phi$ . In other words, amplitude equation (53) is the Euler equation associated with variational principle (52b), arising from variations  $\delta \phi$ .

It is worthwhile to close by noting the differences between the classical approach of the ‘averaged Lagrangian’ and ours. In the former, we start with the Lagrangian density into which we substitute a modal expression of the form (12a). The resultant equation is averaged over the phase, and a suitable variational principle associated with this averaged Lagrangian yields, among other things, an ‘amplitude equation’ (Whitham 1974). On the other hand, we start with a set of differential equations into which we substitute a modal representation to obtain the modal form of the governing equations. A suitable solvability condition applied to these (modal) equations yields the intrinsic Lagrangian. The modes and ‘amplitude equations’ are expressible as the Euler equations of this Lagrangian.

### 8. Discussion and conclusions

We have derived two versions, (32c) and (35), of the equation that governs the (slowly varying) amplitude,  $A(\xi, \tau)$ , of an instability wave packet riding on top of a slightly inhomogeneous base flow. These results were obtained by the use of the high-frequency ansatz, which separates a disturbance into its slowly and rapidly varying parts. The amplitude equation ensures the absence of secular terms at  $O(\epsilon)$ , where  $\epsilon \ll 1$  is the small inhomogeneity parameter. Our results also are interpreted in terms of kinematic wave theories based on various Lagrangian formulations.

What is new in this paper is obviously not the existence of such an amplitude equation, but its very simple form in terms of the components of the displacement variable,  $\alpha$ , in cross- and propagation spaces (i.e.  $Q$  and  $W$ ). We feel that it is reasonable to call  $\mathcal{A}$  the (complex) wave action density: volume integrals (in propagation space) of this quantity are conserved subject to a natural flux and a source term. The latter depends on the changes of the base flow acceleration in the cross-space and vanishes when the base flow is inviscid. Actually, it is only the components of these changes in propagation space that enter into the analysis. The derivation is carried out without any specific restrictions on the dimensionality of the problem.

One major difference between our work and Russell's (1986) is the inclusion of a completely general base flow. Because of this, there will be a source term in the amplitude equation for the unstable wave and a sink term in that for the stable wave. The bilinear wave action density still satisfies a (pure) conservation law, (45c), as in Russell's work, but the individual wave actions do not. Ordinarily, one is interested in the latter in order to calculate the amplitude and the physical characteristics of a packet. These are contained extremely succinctly in our central results (see for example (53)).

Financial help for this work was provided by AFOSR Grant 86-0324. The author gratefully acknowledges the support of Dr James McMichael of AFOSR.

#### REFERENCES

- ANDREWS, D. G. & MCINTYRE, M. E. 1978*a* An exact theory of nonlinear waves on a Lagrangian-mean flow. *J. Fluid Mech.* **89**, 609–646.
- ANDREWS, D. G. & MCINTYRE, M. E. 1978*b* On wave-action and its relatives. *J. Fluid Mech.* **89**, 647–664.
- BERNSTEIN, L., FRIEMAN, E. A., KRUSKAL, M. D. & KULSRUD, R. M. 1958 An energy principle for hydromagnetic stability problems. *Proc. R. Soc. Lond. A* **244**, 17–40.
- BETCHOV, R. & CRIMINALE, W. O. 1967 *Stability of Parallel Flows*. Academic.
- BETCHOV, R. & SZEWCZYK, A. 1963 Stability of a shear layer between parallel streams. *Phys. Fluids* **6**, 1391–1396.
- BRETHERTON, F. P. 1968 Propagation in slowly varying waveguides. *Proc. R. Soc. Lond. A* **302**, 555–576.
- CHIN, W. C. 1980 Effect of dissipation on dispersion of slowly varying wavetrains. *AIAA J.* **18**, 149–158.
- CRIGHTON, D. G. & GASTER, M. 1976 Stability of slowly diverging jet flow. *J. Fluid Mech.* **77**, 397–413.
- DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.
- GASTER, M. 1975 A theoretical model of a wave packet in the boundary layer on a flat plate. *Proc. R. Soc. Lond. A* **347**, 271–289.
- GASTER, M. 1981 On transition to turbulence in boundary layers. In *Transition and Turbulence* (ed. R. Meyer), pp. 95–112. Academic.
- GASTER, M. & GRANT, I. 1975 An experimental investigation of the formation and development of a wave packet in a laminar boundary layer. *Proc. R. Soc. Lond. A* **347**, 253–269.
- GASTER, M., KIT, E. & WYGNANSKI, I. 1985 Large-scale structures in a forced turbulent mixing layer. *J. Fluid Mech.* **150**, 23–29.
- HAYES, W. D. 1968 Energy invariant for geometric acoustics in a moving medium. *Phys. Fluids* **11**, 1654–1656.
- HAYES, W. D. 1970 Conservation of action and modal wave action. *Proc. R. Soc. Lond. A* **320**, 187–208.

- ITOH, N. 1980 Linear stability theory for wave packet disturbances in parallel and nearly parallel flows. In *Proc. IUTAM Symp. on Laminar-Turbulent Transition* (ed. R. Eppler & H. Fasel), pp. 86-95. Springer.
- ITOH, N. 1981 Secondary instability of laminar flows. *Proc. R. Soc. Lond. A* **375**, 565-578.
- LANDAHL, M. T. 1972 Wave mechanics of breakdown. *J. Fluid Mech.* **56**, 775-802.
- LANDAHL, M. T. 1982 The application of kinematic wave theory to wave trains and packets with small dissipation. *Phys. Fluids* **25**, 1512-1516.
- LEWIS, R. M. 1965 Asymptotic theory of wave-propagation. *Arch. Rat. Mech. Anal.* **20**, 191-250.
- LUKE, J. C. 1966 A perturbation method for nonlinear dispersive wave problems. *Proc. R. Soc. Lond. A* **292**, 403-412.
- NAYFEH, A. H. 1980 Three-dimensional stability of growing boundary layers. In *Proc. IUTAM Symp. on Laminar-Turbulent Transition* (ed. R. Eppler & H. Fasel), pp. 201-217. Springer.
- PETERSEN, R. A. & SAMET, M. 1988 On the preferred mode of jet instability. *J. Fluid Mech.* **194**, 153-173.
- RUSSELL, J. M. 1986 Amplitude propagation in slowly varying trains of shear-flow instability waves. *J. Fluid Mech.* **170**, 21-51.
- SELIGER, R. L. & WHITHAM, G. B. 1968 Variational principles in continuum mechanics. *Proc. R. Soc. Lond. A* **305**, 1-25.
- STEWARTSON, K. 1974 Some aspects of nonlinear stability theory. *Fluid Dyn. Trans., Polish Acad. Sci.* **7**, 101-128.
- WHITHAM, G. B. 1965 A general approach to linear and non-linear dispersive waves using a Lagrangian. *J. Fluid Mech.* **22**, 273-283.
- WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Wiley.
- WYGNANSKI, I., CHAMPAGNE, F. & MARASLI, B. 1986 On the large-scale structures in two-dimensional, small-deficit, turbulent wakes. *J. Fluid Mech.* **168**, 31-71.